SOME THEOREMS ON HOLONOMY GROUPS OF RIEMANNIAN MANIFOLDS

BY

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For Riemannian manifolds there are four kinds of holonomy groups: the (nonrestricted) holonomy group H, the restricted holonomy group H^0, the (nonrestricted) homogeneous holonomy group h, and the restricted homogeneous holonomy group h^0. It is known that all of these are Lie groups of transformations and H^0 and h^0 are the connected components of the identity of H and h respectively.

1. **Relations among invariant linear subspaces of h^0 and h.** If the restricted homogeneous holonomy group h^0 is reducible (in the real number field) it is completely reducible, for h is a subgroup of the orthogonal group. If the holonomy group h^0 is reducible, we can take a repère in the tangent space $E_n(O)$ at the base point O of the holonomy group so that all elements of the group h^0 can be represented by matrices of the following type:

$$T = \begin{pmatrix} T_1 & & & & 0 \\ & T_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & T_m \end{pmatrix}.$$

We assume that the group of matrices T_λ is irreducible for each $\lambda(\lambda=1, \cdots, m)$. Let us denote the linear vector space on which T_λ operates by $E_{(\lambda)}$, $E_{(\lambda)}$'s are called irreducible invariant linear subspaces. If T_m is of dimension 1, T_m is equal to 1.

In the same way we can consider the reducibility of the group h. It may happen that, for example, $E_{(1)}$ is not invariant under h although it is invariant under h^0. In such a case there exists a closed curve C_α of class D' passing through the base point 0 of our holonomy groups such that the congruent transformation $T(C_\alpha)$ associated with it takes $E_{(1)}$ into another linear subspace $E_{(1)\alpha}$:

$$T(C_\alpha)E_{(1)} = E_{(1)\alpha}.$$

We shall denote the element of the fundamental group π_1 to which C_α belongs by α.

If we take another closed curve C'_α passing through 0, the product curve $C'_\alpha C_\alpha^{-1} = C_0$ is homotopic to zero. Hence we get

$$T(C_0) = T(C_\alpha^{-1})T(C'_\alpha).$$

As $T(C_\alpha^{-1}) = T(C_\alpha)^{-1}$, we see

$$T(C'_\alpha) = T(C_\alpha)T(C_0),$$

Received by the editors April 6, 1954 and, in revised form, November 15, 1954.

and accordingly we get

$$T(C'_\alpha)E_{(1)} = T(C_\alpha)T(C_0)E_{(1)} = T(C_\alpha)E_{(1)} = E_{(1)\alpha}.$$

Therefore the transformation T associated to any closed curve passing through O and belonging to α takes $E_{(1)}$ to the same $E_{(1)\alpha}$.

In the next place, we can show that the linear subspace $E_{(1)\alpha}$ is invariant under the group h^0. To prove this, let us take an arbitrary closed curve C_0 of class D' passing through O and homotopic to zero. Then the product curve $C_\alpha C_0$ is homotopic to C_α, whence by virtue of the above result we get

$$T(C_0)T(C_\alpha)E_{(1)} = E_{(1)\alpha},$$

whence

$$T(C_0)E_{(1)\alpha} = E_{(1)\alpha},$$

which is to be proved.

In the same way it may happen that transformations associated to closed curves passing through O and belonging to an element β of π_1 take $E_{(1)}$ to a linear subspace $E_{(1)\beta}$ different from $E_{(1)}$ and $E_{(1)\alpha}$. It is easily seen that transformations associated to closed curves passing through O and belonging to the element $\alpha^{-1}\beta$ take $E_{(1)\alpha}$ to $E_{(1)\beta}$.

We shall assume that $\dim(E_{(1)}) \geqq 2$. Then $E_{(1)\alpha}$ must coincide with one of $E_{(2)}, \cdots, E_{(m)}$.

To show this let us take a vector v of $E_{(1)\alpha}$, then it can be written in the following form:

$$v = v_1 + v_2 + \cdots + v_m,$$

where $v_\lambda \in E_{(\lambda)}$. First we assume that $v_1 \neq 0$. According to Borel-Lichnerowicz's theorem(1) we know that the restricted group h^0 is a direct product of component groups of matrices: $h^0 = h^0_{(1)} \times h^0_{(2)} \times \cdots \times h^0_{(m)}$. Hence h^0 contains the group $h^0_{(1)} \times 1 \times \cdots \times 1$ as its subgroup. If we denote a transformation of this group by T, we get

$$T(v) - v = T(v_1) - v_1,$$

the last vector belongs to $E_{(1)} \cap E_{(1)\alpha}$. As $h^0_{(1)}$ is irreducible there exists a T such that $T(v_1) - v_1$ is not the null vector. This contradicts the fact that $E_{(1)} \neq E_{(1)\alpha}$ and $E_{(1)}$ is irreducible. Accordingly v_1 must be equal to zero.

By the same argument, we can see that if $\dim(E_{(\lambda)}) \geqq 2$ (λ fixed), then either $v_\lambda \neq 0$ (other $v_\mu = 0$, $\mu \neq \lambda$) and $E_{(1)\alpha}$ coincides with $E_{(\lambda)}$ or $v_\lambda = 0$. As $E_{(1)\alpha}$ does

(1) A. Borel and A. Lichnerowicz, *Groupes d'holonomie des variétés riemanniennes*, C. R. Acad. Sci. Paris vol. 234 (1952) pp. 1835–1837.

not contain invariant vectors, this proves that $E_{(1)\alpha}$ coincides with one of $E_{(2)}, \cdots, E_{(m)}$. Q.E.D.

We shall change the notation if it is necessary and can assume that $E_{(1)\alpha}$ coincides with $E_{(2)}$.

In the same way, if $E_{(1)\beta}$ is different from $E_{(1)}$ and $E_{(2)}$ it coincides with one of $E_{(3)}, \cdots, E_{(m)}$. We can assume that $E_{(1)\beta}$ coincides with $E_{(3)}$.

Repeating this process we can see that there exists a minimal set of linear subspaces $E_{(1)}, \cdots, E_{(k)}$ such that they are transformed to each other by the holonomy group h and no other linear subspaces are obtained from them by h. Then the direct sum of these spaces constitutes an irreducible invariant subspace of h.

Consequently, we get the following theorem:

THEOREM 1. *Suppose that the restricted homogeneous holonomy group h^0 is reducible and let $E_{(1)}, \cdots, E_{(m)}$ be irreducible invariant subspaces. If $E_{(1)}$ is not invariant under h and* $\dim E_{(1)} \geqq 2$, *we consider the irreducible invariant subspace under h which contains $E_{(1)}$ and denote it by $E^*_{(1)}$. Then we can select, from $E_{(1)}, \cdots, E_{(m)}$, l_1* ($l_1 = \dim E^*_{(1)}/\dim E_{(1)}$) *linear subspaces all of the same dimension such that they span $E^*_{(1)}$ and each of them can be transformed from any other of them by some transformations of h.*

We change the notation if it is necessary and can assume that these l_1 linear subspaces are $E_{(1)}, \cdots, E_{(l_1)}$. If dim $E_{(l_1+1)} \geqq 2$ and $E_{(l_1+1)}$ is not invariant under h, then we can consider the irreducible invariant subspace under h which contains $E_{(l_1+1)}$. We denote it by $E^*_{(2)}$ and assume that $E^*_{(2)}$ consists of $E_{(l_1+1)}, \cdots, E_{(l_1+l_2)}$, and so on.

If there are one-dimensional subspaces among $E_{(1)}, \cdots, E_{(m)}$, we collect them altogether at the last part of the sequence of subspaces. Suppose that $E^{(1)}$ is one of the subspaces $E_{(1)}, \cdots, E_{(m)}$ and such that $\dim E^{(1)}=1$. If $E^{(1)}$ is not invariant under h, then as before there exist vectors $E^{(1)\alpha}, E^{(1)\beta}, \cdots$ invaiiant under h^0 and not equal to $E^{(1)}$ and derived from $E^{(1)}$ by some transformations of h. However, contrary to the former case, we cannot say that $E^{(1)}, E^{(1)\alpha}, E^{(1)\beta}, \cdots$ are orthogonal to each other. We shall investigate in the next section the structure of the transformations of h operating on the irreducible invariant subspace $E^{(1)*}$ containing $E^{(1)}$.

2. **The manifold R^* and its holonomy group H.** Suppose that the holonomy group h of a complete Riemannian manifold M_n is reducible to r- and $(n-r)$-dimensional parts. Then there exist an r dimensional parallel plane field and an $(n-r)$ dimensional parallel plane field orthogonal to each other. Let K be an arbitrary curve in M_n. We denote its initial point by P and its terminal point by Q. We take orthogonal repères $[e_1, \cdots, e_n]_P$ and $[e_1, \cdots, e_n]_Q$ so that their first r vectors $[e_1, \cdots, e_r]_P$ and $[e_1, \cdots, e_r]_Q$ are contained in the r-dimensional planes of the first field at P and Q respectively and the remaining $(n-r)$ vectors $[e_{r+1}, \cdots, e_n]_P$ and $[e_{r+1}, \cdots, e_n]_Q$

are contained in the $(n-r)$-dimensional planes of the other field at P and Q respectively. By developing the curve K in the tangent space $E_n(P)$ with respect to the Euclidean connection of the space, we find a unique image of the point Q and a unique image $[e_1^*, \cdots, e_n^*]$ of the repère $[e_1, \cdots, e_n]_Q$. Then the transformation which takes $[e_1, \cdots, e_n]_P$ to $[e_1^*, \cdots, e_1^*]$ can be expressed by a matrix $T(M_n, K)$ of the type

$$\begin{pmatrix} A & 0 & \alpha \\ 0 & B & \beta \end{pmatrix},$$

where A and B are (r, r) and $(n-r, n-r)$ orthogonal matrices respectively and α and β are $(r, 1)$ and $(n-r, 1)$ matrices respectively. Let us denote by $t(M_n, K)$ the matrix (A, α). The set of all transformations of the type (A, α) constitutes a group.

Let $K=\text{arc } PQ$ be a sufficiently short curve of class D' so that it is contained in a so-called "reduced coordinate neighborhood" such that

$$ds^2 = g_{ab}(x^c)dx^a dx^b + g_{pq}(x^r)dx^p dx^q,$$
$$a, b, c = 1, 2, \cdots, r,$$
$$p, q, r = r+1, \cdots, n.$$

We denote r-dimensional totally geodesic submanifolds belonging to the family $x^p=\text{const.}$ by R and s- $(=n-r)$ dimensional totally geodesic submanifolds belonging to the family $x^a=\text{const.}$ by S, and the R- and S-submanifolds which pass through a point P by R_P and S_P.

We can project K on R_P by using the reduced coordinate neighborhood. We denote the projection by π and denote the image of K by $K'=\pi(K)$. The repère $[e_1, \cdots, e_r]_Q$ impresses a repère $[e_1', \cdots, e_r']_Q$ at the image $Q'=\pi(Q)$ on R_P. By developing the curve K' in the tangent space $E_r(P)$ of R_P with respect to the induced Euclidean connection of the space, we find a unique image $[e_1'^*, \cdots, e_r'^*]$ of the repère $[e_1', \cdots, e_r']_Q$. Let us denote the transformation which takes $[e_1, \cdots, e_r]_P$ to $[e_1'^*, \cdots, e_r'^*]$ by $T(R_P, K')$. Then

LEMMA 1. $t(M_n, K)=T(R_P, K')$.

If we write down the equations of definition of the Euclidean connection using a reduced coordinate neighborhood which contains K we can easily see the truth of our assertion, for in this case the equations of definition of the connection separate into two parts corresponding to R- and S-submanifolds.

Let R_a and R_b be two R-submanifolds and assume that $P\in R_a$ and $Q\in R_b$ lie on the same S-submanifold. By a R-neighborhood of P we mean a neighborhood of P in R_P. Then there exists an isometry χ between some R-neighborhoods $V(P)$ on R_a and $V(Q)$ on R_b such that corresponding points on R_a and R_b lie on the same S-submanifolds.

To prove this we connect P and Q by a geodesic $[PQ]$ in the S-manifold in which P and Q are. Divide $[PQ]$ so fine that if we denote the dividing points by $P=z_0, z_1, \cdots, z_k=Q$, the reduced neighborhood $U(z_i)$ of z_i contains z_{i+1}. Then it is clear that there arises a sequence of isometries between suitable R-neighborhoods $V(z_i)\subset U(z_i)\cap R_{z_i}$ $(i=0, 1, \cdots, k)$.

In the next place, let L be a curve on R_a passing through P. Then we can prolong the isometry χ of $V(P)$ and $V(Q)$ along L on R_a and its image on R_b, a point on R_a and its image on R_b lying always on the same S-submanifold.

To prove this we first take a point P_1 on the connected component of $P_0\equiv P$ on $V(P)\cap L$ and draw its image curve and denote the image of P_1 by Q_1. Draw the geodesic segment $[P_1Q_1]$ near $[PQ]$ and starting from $[P_1Q_1]$ we can construct an isometry between some R-neighborhoods $V(P_1)$ on R_a and $V(Q_1)$ on R_b. It is evident that on $V(P)\cap V(P_1)$ and $V(Q)\cap V(Q_1)$ both isometries are identical and hence we can prolong the isometry $V(P)\rightleftarrows V(Q)$ to the isometry $V(P)\cup V(P_1)\rightleftarrows V(Q)\cup V(Q_1)$. Repeating this reasoning we can easily see that our assertion is true because we can choose a sequence of new neighborhoods so that the diameters of the new neighborhoods which arise successively by analogous construction have a positive lower bound. We denote the prolonged isometry also by χ.

Let us denote the image of L on R_b by M. We impress the repère $[e_1, \cdots, e_r]_P$, at the initial point P of L spanning the r-dimensional tangent plane of R_a at P to the initial point Q of M and likewise impress the repère $[e_1, \cdots, e_r]_{P'}$ at the terminal point P' of L spanning the r-dimensional tangent plane of R_a at P' to the terminal point Q' of M. Then

Lemma 2. $T(R_a, L)=T(R_b, M)$.

Now let us consider a closed curve C passing through the base point O of the holonomy group. We divide the curve C by m points $O\equiv P_0, P_1, \cdots, P_{m-1}, P_m\equiv O$ so fine that the subarc $P_\lambda P_{\lambda+1}$ is contained in a reduced coordinate neighborhood $U(P_\lambda)$ for every $\lambda(\lambda=0, 1, \cdots, m-1)$. We take at $P_0=P_m$ and $P_1, \cdots, P_{m-1}$ repères so that their first r vectors $[e_1, \cdots, e_r]$ span r-dimensional tangent planes at them. And we consider the development of C in the tangent space $E_n(P_0)$. Then, we get the following relation.

Lemma 3. *$t(M_n, C)=T(R_{P_0}, C')$, where C' is the continuous projection of C by π such that $\pi(P_0)=P_0$.*

Proof. First we get by virtue of Lemma 1

$$t(M_n, C) = T(R_{P_{n-1}}, \text{arc } P_{m-1}P'_m)t(M_n, \text{arc } P_0P_{m-1} \text{ of } C),$$

where arc $P_{m-1}P'_m$ is the image of the arc $P_{m-1}P_m$ by the projection π. The right-hand side of the last equation is, by virtue of Lemmas 1 and 2, equal to the following transformation:

$$T(R_{P_{m-2}}, \text{arc } P_{m-2}P''_m)t(M_n, \text{arc } P_0P_{m-2} \text{ of } C),$$

where the arc $P_{m-2}P''_m$ is the image of (arc $P_{m-2}P_{m-1}$ of C)+arc $P_{m-2}P'_{m-1}$ by the projection π and isometry χ on $R_{P_{m-1}}$. Iterating this process we can see that Lemma 3 is true.

We are now going to assume a hypothesis(2).

HYPOTHESIS W. There exists in M_n a point O such that each point of the submanifold R_0 has an R-neighborhood which meets at most once with any S-submanifold.

When this hypothesis is satisfied, we take such a point O as the base point of holonomy groups. Then, for any S-submanifold, the intersection $S \cap R_0$ is a discrete set of points. We say that any two points of this set are congruent to each other. Then a sufficiently small R-neighborhood of a point on R_0 is isometric with corresponding R-neighborhoods of its congruent points. Hence if we identify congruent points on R_0, there arises a manifold R^* such that R_0 is a covering manifold of R^*. As M_n is assumed to be complete, R_0 and R^* are also complete. The terminal point O' of the curve C' does not in general coincide with the point O, except in the case when C is homotopic to zero. However, by the construction, O' is congruent to O. Hence the image C^* of C' on R^* is a closed curve passing through O^* (image of O) on R^*. As R_0 and R^* correspond locally isometrically we can easily see that

$$T(R_0, C') = T(R^*, C'^*) \in H(R^*),$$

where $H(R^*)$ means the holonomy group of R^*.

Conversely, let us consider a closed curve of class D' passing through the base point O^* and the transformation $T(C^*)$ of the holonomy group $H(R^*)$ associated with C^*. As R_0 is a covering manifold of R^* we can construct the curve C' which issues from the point O over O^* and lies over C^*. The terminal point O' of C' is a point which is congruent to O. We can easily see that

$$T(C^*) = T(R_0, C'),$$

provided that the repères $[e_1, \cdots, e_r]_O$ and $[e_1, \cdots, e_r]_{O'}$ at the initial and terminal points of C' are those which lie over the repère at the point O^*.

Now, as O and O' lie on the same submanifold S_0 we can connect O' with O by a curve C'' of class D' on S_0. If we consider the product curve $C'C''$ as a curve in M_n, the continuous projection of $C'C''$ into R_0 is easily seen to be C'. The repère $[e_1, \cdots, e_r]_{O'}$ we mentioned above is also the image of $[e_1, \cdots, e_r]_O$ by the projection. Hence we can see that

$$T(R_0, C') = t(M_n, C'C'').$$

Consequently, we get the following

THEOREM 2. *Let M_n be a complete Riemannian manifold whose holonomy group h decomposes in r-dimensional and $(n-r)$-dimensional parts and satisfies the hypothesis* W. *We take a point O satisfying the hypothesis* W *and construct*

(2) A. G. Walker, *The fibering of Riemannian manifolds*, Proc. London Math. Soc. (3) vol. 3 (1953) pp. 1–19.

the submanifold R_0 and the manifold R^* which arises by identification of congruent points of R_0. Then the group which consists of all transformations $t(M_n, C)$ is the same as the holonomy group $H(R^*)$ of the manifold R^*, the base point O^* and the repère at O^* on R^* being naturally impressed from M_n.

Suppose that the restricted holonomy group h^0 of M_n decomposes and fixes r vectors (we do not assume that there are no other invariant vectors), and that these r vectors span an r-dimensional plane invariant under the holonomy group h. Then there exist a parallel field of r-dimensional planes in M_n and $(n-r)$-parameter family of r-dimensional totally geodesic submanifolds R. Each of these submanifolds R is an Euclidean space form, in other words, a complete manifold with locally flat Riemannian metric. Hence the manifold R^* is also a Euclidean space form. However, as is known, "the universal covering manifold of any Euclidean space form of n dimensions is the n-dimensional Euclidean space E_n and the holonomy group H of it coincides with the group of covering transformations on E_n". Hence, $H(R^*)$ is nothing but a discrete group of congruent transformations without fixed points of E_n. Accordingly we get, by virtue of Theorem 2, the following

THEOREM 3. *Suppose that the holonomy group h^0 of a complete Riemannian manifold M_n decomposes and fixes r vectors and that these r vectors span an r-dimensional plane invariant under the holonomy group h. If it moreover satisfies the hypothesis* W, *the r-dimensional part corresponding to the invariant r-dimensional plane of the holonomy group $H(M_n)$ is a discrete group of congruent transformations without fixed points of E_n.*

REMARK. We can easily see from the fact " $\cdots$ " cited above that a (non-restricted) homogeneous holonomy group h is not always closed in the orthogonal group $O(n)$. For example, consider the cyclic group generated by the following transformation of E_n

$$\begin{aligned} x_1' &= \cos \alpha x_1 - \sin \alpha x_2, \\ x_2' &= \sin \alpha x_1 + \cos \alpha x_2, \qquad \alpha/\pi \text{ irrational}, \\ x_3' &= x_3 + 1. \end{aligned}$$

The factor space of E_n by this group has obviously the desired property.

3. A theorem on the group H^0.

THEOREM 4(3). *Let M_n be an irreducible Riemannian manifold. Then the holonomy group H^0*

(i) *either contains all translations of Euclidean space E_n*

(ii) *or it fixes a point in E_n (in other words, it is a subgroup of the rotation group $O^+(n)$ with a center at the fixed point).*

(3) We owe this theorem to A. Borel. But for the sake of completeness we shall write our proof here.

Proof. We shall indicate an element of H^0 considered as a topological group by g and the motion associated with g by

$$T(g): x_i' = a_{ij}(g)x_j + a_i(g). \tag{1}$$

Then all the transformations $T(g)$ constitute H^0. Of course, the matrices

$$A(g) = (a_{ij}(g))$$

are nothing but the coefficient matrices of transformations of h^0 and this set is irreducible by our assumption.

Now let us consider the representation $\Gamma: g \rightarrow A(g)$ and denote the kernel of Γ by K. Then K is the totality of elements of H^0 such that $A(g) = E$, hence K is the subgroup of H^0 consisting of all translations of H^0. We shall classify two cases; the first is the case where K is nondiscrete and the second is the case where K is discrete.

(i) *The case where K is a nondiscrete group.* As K is a closed subgroup of H^0, K is a Lie group. Hence K contains at least a one parameter group K_1 of translations as its subgroup. Let us denote it by

$$\Lambda_t: x_i' = x_i + \lambda_i t \tag{2}$$

where λ_i are constant such that at least one of them is not equal to zero. Now denoting an arbitrary element of H^0 by $T(g)$ we can easily verify that $T(g)\Lambda_t T(g)^{-1}$ is a translation and its equation is given by

$$x_i' = x_i + a_{ih}(g)\lambda_h t. \tag{3}$$

As the set of matrices $(a_{ij}(g))$ is irreducible, we see immediately that K contains n linearly independent translations. Hence K contains all translations of E_n.

(ii) *The case where K is a discrete group.* As K is the kernel of the representation Γ, K is a normal subgroup of H^0. Hence, by virtue of the theorem(4) to the effect that every discrete normal subgroup of a connected topological group is a central normal subgroup of this group, K is contained in the center of H^0. Accordingly, if we assume that $T(g)$ and $\Lambda: x_i' = x_i + \lambda_i$ are transformations belonging to H^0 and K respectively, then $T(g)\Lambda T(g)^{-1}$ must coincide with Λ. Hence, we can see that the equation

$$\lambda_i = a_{ik}(g)\lambda_k$$

must hold for every $g \in H^0$ and fixed λ_i. If there is one λ which is not equal to zero among λ_i, the last equation shows that there exists at least an invariant direction under h^0 which contradicts the fact that h^0 is irreducible by our assumption. Accordingly, every λ_i must vanish and hence K consists only of the identity. Consequently, we can conclude that the representation Γ is faithful, in other words there exists an isomorphism $H^0 \cong h^0$.

(4) Pontrjagin, *Topological groups*, p. 77.

By virtue of a theorem of Borel-Lichnerowicz[5], the group h^0 is a closed subgroup of the compact orthogonal group $O^+(n)$ and hence h^0 is compact. As $H^0 \cong h^0$, H^0 is also compact. Accordingly, we can introduce in the group manifold of H^0 the Haar measure. Denoting the total measure of the group manifold by ω, we put

$$\frac{1}{\omega}\int a_i(g)dg = c_i.$$

In the last and the following equations we assume that the integrals are extended over the whole group manifold. Then we can easily see that

$$a_{ik}(g)c_k + a_i(g) = \frac{1}{\omega}\int (a_{ik}(g)a_k(h) + a_i(g))dh.$$

As the integrand of the right-nand side of the last equation is equal to $a_i(hg)$, we get

$$a_{ik}(g)c_k + a_i(g) = \frac{1}{\omega}\int a_i(hg)dh.$$

Since the Haar measure over the compact group is two-sided invariant, the right-hand side is equal to

$$\frac{1}{\omega}\int a_i(h)dh = c_i.$$

Consequently, c_i is the fixed point under the group H^0. Thus our theorem is completely proved.

4. A theorem on complete Riemannian manifolds.

THEOREM 5. *Let M_n be a complete Riemannian manifold. If the holonomy group H^0 fixes a point, then M_n is an Euclidean space form.*

Proof. We shall prove the theorem under the assumption that M_n is simply connected. However, if this is done, the general case follows immediately. For, as the holonomy group $\tilde{H} = \tilde{H}^0$ of the universal covering manifold $\tilde{M}_n$ of M_n with naturally induced matric from M_n coincides with H^0, $\tilde{M}_n$ is an Euclidean space form by hypothesis and hence M_n itself is everywhere locally Euclidean.

Let us denote the base point of the holonomy group by O and denote the fixed point in the tangent space $E_n(O)$ by P_0^*. Then, we can draw a geodesic segment arc OP_0 in M_n such that its development in $E_n(O)$ coincides with the straight line OP_0^* by virtue of the completeness assumption. We shall consider a normal coordinate system with center P_0 and denote it by $\bar{x}$ and the

[5] A. Borel and A. Lichnerowicz, loc. cit.

coordinate neighborhood by U. It is well known that

$$\left\{\begin{matrix}\bar{i}\\ jk\end{matrix}\right\}\bar{x}^j\bar{x}^k = 0 \tag{4}$$

holds good in U.

Now let us consider a point $P-\bar{x}^ie_i$ at every tangent space $E_n(P)$ $(P\in U)$ on a geodesic $\bar{x}^i=a^is$ through P_0, where we assume that e_i is the natural repère at that point. Then, by virtue of (4) we get

$$\frac{d}{ds}(P - \bar{x}^ie_i) = \left(\frac{d\bar{x}^i}{ds} - \frac{d\bar{x}^i}{ds} - \bar{x}^j\left\{\begin{matrix}\bar{i}\\ jk\end{matrix}\right\}a^k\right)e_i = 0.$$

Hence, the point $P-\bar{x}^ie_i\in E_n(P)$ at each point P of the geodesic $\bar{x}^i=a^is$ concides when we develop these tangent spaces along the geodesic. If we consider the case $\bar{x}^i=0$ we can see that the point $P-\bar{x}^ie_i$ coincides with the invariant point P_0^* of the holonomy group H^0. Hence $P-\bar{x}^ie_i$ in $E_n(P)$ is the point which is transplanted from P_0^* by the connection of the manifold M_n, and it does not depend upon the curves which combine P_0 to P.

Let us now consider another curve C through P and consider the derivative of $P-\bar{x}^ie_i$ with respect to this curve. As $P-\bar{x}^ie_i$ is a covariant constant point field, we get $(d/ds)(P-\bar{x}^ie_i)=0$, which reduces to

$$\left\{\begin{matrix}\bar{i}\\ jk\end{matrix}\right\}\bar{x}^j\frac{d\bar{x}^k}{ds}e_i = 0.$$

As the curve C may have any direction at P, we get from the last equation the following relation:

$$\left\{\begin{matrix}\bar{i}\\ jk\end{matrix}\right\}\bar{x}^j = 0.$$

From the last relation we get easily

$$(\partial\bar{g}_{jk}/\partial\bar{x}^i)\bar{x}^i = 0,$$

which shows that $\bar{g}_{jk}(\bar{x})$'s are homogeneous functions of degree 0 with respect to x^i[6]. Hence, as P_0 is a regular point of the manifold M_n, we see that $\bar{g}_{jk}(\bar{x})=\bar{g}_{jk}(0)$ in the neighborhood U. Accordingly, in the domain U our manifold is locally Euclidean.

In the next place we shall show that any two geodesic rays which issue from O do not intersect any more. By hypothesis M_n is simply connected. Hence the point P_0^* can be transplanted uniquely on every tangent space $E_n(P)$ $(P\in M_n)$ by development irrespective to curves which bind P_0 to P. It is

[6] S. Tachibana, *On the normal coordinate of Riemann space, whose holonomy group fixes a point*, Tôhoku Math. J. (2) vol. 1 (1949) pp. 26–30.

evident that the transplanted point does not coincide with P unless P coincides with P_0. Now, let us assume that there exist two geodesic rays which intersect at $Q(\neq P)$ and denote them by g_1 and g_2. If we consider the points R_1^*, R_2^* on $E_n(Q)$ which lie on the tangents of g_1 and g_2 at Q such that QR_λ^* = the length of arc QP_0 along g_λ $(\lambda=1, 2)$, then R_1^*, R_2^* are the transplanted points of P_0^* along g_λ. This contradicts the fact that P_0^* is transplanted uniquely irrespective to curves which bind P_0 to Q. Accordingly, any two geodesic rays which issue from P_0 do not intersect any more. In other words, geodesic rays which issue from P_0 constitute a geodesic field in the large.

Now let us denote by K_l the inner domain of the geodesic hypersphere of radius l and with center P_0 and by ∂K_l its boundary. We take l so large that every point $P \in K_l$ has a locally flat neighborhood but some points on ∂K_l do not have such property. If $l=\infty$, then our theorem is proved, so we assume that l is finite and $Q \in \partial K_l$ is one of the points which do not have the above stated property.

As the group H^0 fixes the point P_0^*, there exists a neighborhood V of Q such that the line element ds^2 in V can be written as(7)

$$ds^2 = (x^n)^2 g_{ab}(x^c)dx^a dx^b + (dx^n)^2 \qquad a, b, c = 1, 2, \cdots, n-1.$$

If a point $R \in V$ has the coordinates (x^a, x^n), then x^n = the length of the geodesic segment P_0R and the geodesic hyperspheres x^n = const. are umbilical hypersurfaces too. However, when $x^n < l$ the line element is locally flat by assumption, hence it is also locally flat for $x^n \geqq l$. Hence the line element is locally flat in V. This contradicts the fact that Q has no locally flat neighborhood. Accordingly $l=\infty$ i.e. M_n is locally Euclidean everywhere. Q.E.D.

COROLLARY 1. *Let M_n be a complete Riemannian manifold which is irreducible (with respect to the holonomy group h^0). Then the holonomy group H^0 contains all translations of the Euclidean space E_n.*

COROLLARY 2. *Let M_n be a complete Riemannian manifold. Then the holonomy group H^0 is a closed subgroup of the group of motions.*

COROLLARY 3. *Let M_n be a complete and simply connected Riemannian manifold. If its nonhomogeneous holonomy group H^0 fixes an r-dimensional linear subspace E_r $(0<r<n)$ of E_n, then*

$$V_r = V_x \times E_{n-r}.$$

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(7) S. Sasaki, *On the structure of Riemannian spaces whose holonomy groups fix a direction or a point*, Journ. Physico Math. Soc. Japan vol. 16 (1942) pp. 193–200 (in Japanese).